

Cross-section Lattices of \mathcal{J} -irreducible Monoids and Orbit Structures of Weight Polytopes *

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Abstract

Let λ be a dominant weight of a finite dimensional simple Lie algebra and W the Weyl group. The convex hull of $W\lambda$ is defined as the weight polytope of λ . We provide a new proof that there is a natural bijection between the set of orbits of the nonempty faces of the weight polytope under the action of the Weyl group and the set of the connected subdiagrams of the extended Dynkin diagram that contain the extended node $\{-\lambda\}$. We show that each face of the polytope can be transformed to a standard parabolic face. We also show that a standard parabolic face is the convex hull of the orbit of a parabolic subgroup of W acting on the dominant weight. In addition, we find that the linear space spanned by a face is in fact spanned by roots.

Keywords: Weight polytope, Parabolic face, Cross-section lattice, Weyl group, Dynkin diagram.

2010 Mathematics Subject Classification: 17B45, 17B67, 20M32.

1 Introduction

The interests of studying the set of orbits of the faces of the weight polytopes under the action of the Weyl groups arise partially from the linear algebraic monoid theory [13, 14, 16]. A linear algebraic monoid is *irreducible* if it is irreducible as a variety. An irreducible monoid is *reductive* if its unit group is reductive. Let M be a reductive monoid with unit group G . Consider the group action of $G \times G$ on M by $(g, h) \cdot a = gah^{-1}$ for $g, h \in G$ and $a \in M$. The set $G \backslash M / G$ of orbits GaG for this action is a poset with respect to inclusion. Associated with a reductive monoid is a polytope on which the Weyl group W acts naturally. The lattice of orbits of M is isomorphic to the lattice of orbits of the faces of the polytope under the action of W . An interesting problem in linear algebraic monoid theory is to find the orbits of the reductive monoid M . In general, the problem is still open.

*Project supported by national NSF of China (No 11171202).

Let B be a Borel subgroup of G with maximal torus $T \subseteq B$. Denote by \overline{T} the Zariski closure of T in M , and let $E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}$. The *cross-section lattice* of M by definition is

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\},$$

which is isomorphic to $G \backslash M / G$. In particular,

$$M = \bigsqcup_{e \in \Lambda} GeG \quad (\text{disjoint union}).$$

It is interesting and challenging to find the cross-section lattice combinatorially. If M has zero and its cross-section lattice has a unique minimal nonzero element, then M is referred to as \mathcal{J} -irreducible. The cross-section lattices of \mathcal{J} -irreducible monoids are described explicitly in [15]. In particular, let G_0 be a semisimple algebraic group and $\rho : G_0 \rightarrow GL(V)$ be a complex irreducible rational representation with highest weight λ . Then $G = K^*\rho(G_0)$ is reductive and

$$M(\lambda) = \overline{G} \quad (\text{Zariski closure in } \text{End}(V))$$

is a \mathcal{J} -irreducible monoid and all \mathcal{J} -irreducible monoids can be constructed this way up to a finite morphism. Let \mathfrak{g} be the Lie algebra of G and Φ be its root system with a root basis $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Denote by E the Euclidean space spanned by Π , and by $\{\mu_1, \dots, \mu_n\}$ the fundamental dominant weights with respect to Π . If $\lambda = a_1\mu_1 + \dots + a_n\mu_n$, the set

$$J = \{j \in \mathbf{n} \mid a_j = 0\}$$

is called the *type of $M(\lambda)$* or simply the *type of λ* where $\mathbf{n} = \{1, \dots, n\}$. The result below describing the cross-section lattices of \mathcal{J} -irreducible monoids is a summary of Corollary 4.11 and Theorem 4.16 of [15].

Theorem A. *Let J be the type of $M(\lambda)$ with λ dominant. Then the nonzero elements of the cross-section lattice of $M(\lambda)$ are in bijection with the subsets I of Π where no connected component of I lies entirely in J .*

The cross-section lattice of $M(\lambda)$ where the dominant weight λ is the highest root of \mathfrak{g} was explicitly described in [9, Theorem 3.1] using connected subsets of the affine Dynkin diagram.

The cross-section lattice of the \mathcal{J} -irreducible monoid $M(\lambda)$ can be characterized using polytopes [13, 14, 16]. The polytope associated with $M(\lambda)$ is the convex hull of the orbit $W\lambda$, called the *weight polytope of $M(\lambda)$* , or simply the *weight polytope of λ* . If λ is the highest root of \mathfrak{g} then the weight polytope is a *root polytope*. The following result can be deduced from [8, Theorem 2] and [13, Theorem 8.4] and [16, Section 5].

Theorem B. *There is a one to one correspondence between the cross-section lattice of $M(\lambda)$ and the set of orbits of the faces of the weight polytope under the action of the Weyl group.*

The weight polytope of λ depends only on the type J of λ , so we call J the type of the weight polytope. All the strongly dominant weights are of the same type $J = \emptyset$, the empty set.

Theorem C. *If the type of the weight polytope is J , then the set of orbits of the nonempty faces of the weight polytope is in bijection with the set of subsets I of Π where no connected component of I lies entirely in J .*

Though the theory of algebraic monoids accelerated in the last three decades, some researchers do not know that the subject exists, and based on some minimal evidence, some are discouraged by prerequisites which seem a little hard to defeat.

In this paper, we restate Theorem C in terms of the language of Lie algebras and prove it using techniques from the theory of Lie algebras to make it more accessible to Lie algebraists. Adding a new node $\{-\lambda\}$ to the Dynkin diagram of the Lie algebra \mathfrak{g} by drawing a single laced edge between $\{-\lambda\}$ and α_i if $(\lambda, \alpha_i) > 0$ where $(\ , \)$ is the inner product on the Euclidean space E , we obtain an *extended Dynkin diagram*. Theorem C can then be described in Theorem D, which is a consequence of our Theorems 4.5 and 4.15.

Theorem D. *The orbits of the nonempty faces of the weight polytope under the action of the Weyl group are in bijection with the connected subdiagrams of the extended Dynkin diagram that contain the extended node $\{-\lambda\}$.*

If λ is the highest root of the Lie algebra \mathfrak{g} , then Theorem D specializes the bijection for root polytopes proved in [3, Theorem 5.9], which makes use of the properties of the root system of \mathfrak{g} . The methods in [3] motivate us to prove Theorem D without using the theory of linear algebraic monoids. Our arguments are based on the weight set of \mathfrak{g} , which is lacking of some advantages of the root system, so we need new techniques to overcome this difficulty. This is reflected in the proofs of the results in Section 4. We notice that different researchers study algebraic, geometrical, and combinatorial properties of root polytopes, and we refer the reader to [1, 4, 10, 11] for more details.

Our first main result Theorem 4.3 states that a standard parabolic face is the convex hull of the orbit of a parabolic subgroup of W acting on the dominant weight. This leads to the proof of Theorem 4.5. The second main result Theorem 4.14 is an analogue of [3, Corollary 5.4] for weight polytopes, stating that the linear space spanned by a face of a weight polytope is in fact spanned by roots. This result is used to prove that all nonempty faces of a weight polytope are parabolic in Theorem 4.15.

The rest of the paper is organized as follows. Section 2 provides some necessary background knowledge. Section 3 is a summary of notation and terminology. Section 4 is dedicated to the proof of the main results of the paper.

2 Preliminaries

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra. We fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and assume that Φ is the root system of \mathfrak{g} with respect to \mathfrak{h} . Then \mathfrak{g} has the following root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

Let $\mathbf{n} = \{1, 2, \dots, n\}$, $\Pi = \{\alpha_i \mid i \in \mathbf{n}\}$ be a root basis of Φ , and W be the Weyl group generated by the simple reflections $\{s_{\alpha_i} \mid i \in \mathbf{n}\}$. Let $\{\omega_i \mid i \in \mathbf{n}\}$ be the fundamental weights with respect to Π , and let $\tilde{\omega}_i = \frac{2\omega_i}{(\alpha_i, \alpha_i)}$ be the coweights of ω_i . We have $(\tilde{\omega}_i, \alpha_j) = \delta_{ij}$ for $i, j \in \mathbf{n}$.

Denote by $P(\lambda)$ the set of weights of the irreducible highest weight module with highest dominant weight λ . Then $P(\lambda)$ is saturated, and there is a partial order on $P(\lambda)$: $\mu \leq \nu$ if $\nu - \mu$ is a sum of simple roots. We will use repeatedly the following known results in the representation theory of Lie algebras.

Proposition 2.1 ([6, Proposition 3.6]) *For $\mu \in P(\lambda)$ and $\alpha_i \in \Pi$, suppose that the α_i -string through μ is: $\mu - p\alpha_i, \dots, \mu, \dots, \mu + q\alpha_i$. Then $p - q = \frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)}$. In particular, if $(\mu, \alpha_i) > 0$, then $\mu - \alpha_i \in P(\lambda)$.*

Proposition 2.2 ([2, Chapter VI, Proposition 24]) *Let Φ be a root system and let Φ' be the intersection of Φ with a subspace of E . Then Φ' is a root system in the subspace spanned by Φ' .*

Let $I \subseteq \mathbf{n}$ and W_I be the parabolic subgroup of W generated by s_{α_i} for $i \in I$. Let W^I be the minimal length representatives of left cosets of W_I in W . Then

$$\begin{aligned} W^I &= \{w \in W \mid w \cdot \alpha_i > 0 \text{ for } i \in I\} \\ &= \{w \in W \mid l(ws_{\alpha_i}) = l(w) + 1 \text{ for } i \in I\}. \end{aligned}$$

Proposition 2.3 ([7, Chapter 5, Lemma C]) *Every element $w \in W$ has a unique decomposition $w = w^I w_I$ such that $l(w) = l(w^I) + l(w_I)$, where $w^I \in W^I$, $w_I \in W_I$.*

3 Notation and Terminology

Fix a dominant weight λ in the euclidean space E spanned by Π , once and forever. The convex hull \mathbf{P} of $P(\lambda)$ is a polytope equal to the convex hull of $W\lambda$.

Definition 3.1 *We call \mathbf{P} the weight polytope of λ .*

Let $\lambda = \sum_{i=1}^n m_i \alpha_i$. Then all the coefficients m_i 's are non-negative rational numbers. Without loss of generality, we may assume that all m_i 's are non-negative integers since a dilation of a polytope and itself have the same face lattice structure. The following is a consequence of Proposition 11.3 of [6].

Proposition 3.2 *If all the coefficients in $\lambda = \sum_{i=1}^n m_i \alpha_i$ are integers, then $P(\lambda)$ is the intersection of the weight polytope \mathbf{P} with the root lattice.*

If we express a weight μ as a linear combination of simple roots α_i , we use $c_i(\mu)$ to denote the coefficient of α_i . For example, $c_i(\lambda) = m_i$. Let P_i be the set of weights μ such that $(\mu, \tilde{\omega}_i) = m_i$, in other words,

$$P_i = \{\mu \in P(\lambda) \mid c_i(\mu) = m_i\}.$$

Definition 3.3 *The convex hull of P_i is called the i th coordinate face of \mathbf{P} , and will be denoted by F_i .*

Each coordinate face F_i is a face of the weight polytope \mathbf{P} since F_i sits in the hyperplane $\{x \in E \mid (x, \tilde{\omega}_i) = m_i\}$ and all other weights in F_i are in the half-space $\{x \in E \mid (x, \tilde{\omega}_i) \leq m_i\}$. It is, however, possible that F_i is not a facet.

Definition 3.4 *A face of \mathbf{P} is standard parabolic if it is an intersection of coordinate faces; a face of \mathbf{P} is parabolic if it can be transformed into a standard parabolic face by an element of W .*

For any $I \subseteq \mathbf{n}$, the face

$$F_I = \{\mu \in \mathbf{P} \mid c_i(\mu) = m_i \text{ for } i \in I\}$$

is standard parabolic, since $F_I = \cap_{i \in I} F_i$. On the other hand, for any standard parabolic face F , there exists $I \subseteq \mathbf{n}$ such that $F = F_I$. Clearly,

$$P_I = \cap_{i \in I} P_i = \{\mu \in P(\lambda) \mid c_i(\mu) = m_i \text{ for } i \in I\}$$

is the set of all weights in F_I . Note that $F_{\mathbf{n}} = P_{\mathbf{n}} = \{\lambda\}$. For convenience, let $F_{\emptyset} = \mathbf{P}$ and $P_{\emptyset} = P(\lambda)$. So the standard parabolic face F_I is not empty for all $I \subseteq \mathbf{n}$.

Definition 3.5 *An extended Dynkin diagram is the diagram obtained by adding a new node $\{-\lambda\}$ to the Dynkin diagram, in which a single laced edge is drawn between $\{-\lambda\}$ and α_i if and only if $(\lambda, \alpha_i) > 0$ for $i \in \mathbf{n}$.*

4 Weight Polytope

The purpose of this section is to show in Theorem 4.5 that there is a bijection between the set of standard parabolic faces and the set of certain connected components in the extended Dynkin diagram, and then in Theorem 4.15 that all nonempty faces of a weight polytope are parabolic.

If I is a subset of \mathbf{n} , let \bar{I} be the complement set of $I \subseteq \mathbf{n}$, and let $\Pi_I = \{\alpha_i \in \Pi \mid i \in I\}$. We begin our discussion by introducing the following two lemmas.

Lemma 4.1 *Let μ, ν be weights in a standard parabolic proper face F_I of \mathbf{P} . If $\mu < \nu$ then there are simple roots $\alpha_{i_1}, \dots, \alpha_{i_k} \in \Pi_{\bar{I}}$ such that $\nu = \mu + \alpha_{i_1} + \dots + \alpha_{i_k}$ and all $\mu + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, 2, \dots, k$.*

Proof. Let $\nu - \mu = \sum_{j \in J} a_j \alpha_j$ where $J \subseteq \mathbf{n}$ and a_j are positive integers for all $j \in J$. We claim that $I \cap J = \emptyset$ and $(\nu - \mu, \alpha_{j_0}) > 0$ for some $j_0 \in J$. Otherwise, if $i \in I \cap J$, then $c_i(\mu) = m_i - a_i < m_i$, which contradicts that $\mu \in F_I$. If $(\nu - \mu, \alpha_j) \leq 0$ for all $j \in J$, then $(\nu - \mu, \nu - \mu) = 0$, and hence $\nu = \mu$, which is a contradiction.

Use induction on the height of $\nu - \mu$: $\text{ht}(\nu - \mu) = \sum_{j \in J} a_j$. If $\text{ht}(\nu - \mu) = 1$, then the result is true. Assume the result in Lemma 4.1 is true for $\text{ht}(\nu - \mu) = k - 1$ where $k > 1$. We show that the result is true for $\text{ht}(\nu - \mu) = k$ case by case.

Case 1: $(\nu, \alpha_{j_0}) > 0$.

It follows from Proposition 2.1 that $\nu - \alpha_{j_0}$ is a weight in F_I , and $\mu < \nu - \alpha_{j_0}$ with $\text{ht}(\nu - \alpha_{j_0} - \mu) = k - 1 > 0$. Applying the induction hypothesis, we have $\nu - \alpha_{j_0} = \mu + \alpha_{i_1} + \dots + \alpha_{i_{k-1}}$ such that all $\mu + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, 2, \dots, k - 1$. Let $\alpha_{i_k} = \alpha_{j_0}$. Then $\nu = \mu + \alpha_{i_1} + \dots + \alpha_{i_{k-1}} + \alpha_{i_k}$ and all $\mu + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, 2, \dots, k$.

Case 2: $(\nu, \alpha_{j_0}) \leq 0$.

In this case, $(\mu, \alpha_{j_0}) < (\nu, \alpha_{j_0}) \leq 0$. Then $\mu + \alpha_{j_0}$ is a weight in F_I by Proposition 2.1. Therefore, $\text{ht}(\nu - (\mu + \alpha_{j_0})) = k - 1 > 0$ and $\mu + \alpha_{j_0} < \nu$. The induction hypothesis shows that $\nu = (\mu + \alpha_{j_0}) + \alpha_{i_1} + \dots + \alpha_{i_{k-1}}$ where all $(\mu + \alpha_{j_0}) + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, 2, \dots, k - 1$. Reorder the indices by setting $\alpha_{i_1} = \alpha_{j_0}$ and $\alpha_{i_t} = \alpha_{i_{t-1}}$ for $t = 2, \dots, k$. Then we have $\nu = \mu + \alpha_{i_1} + \dots + \alpha_{i_k}$ and all $\mu + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, 2, \dots, k$. \square

The following lemma is easy to understand by intuition, however, its proof is not trivial and the authors don't find it in any existing literature.

Lemma 4.2 Assume that $w\lambda = \lambda - \sum_{j \in J} a_j \alpha_j$ where $w \in W$, a_j are positive integers, and $\alpha_j \in \Pi$ for all $j \in J$. Then there exists $u \in W_J$ such that $w\lambda = u\lambda$.

Proof. By Proposition 2.3, $w = w^J w_J$ where $w^J \in W^J$ and $w_J \in W_J$ with $l(w) = l(w^J) + l(w_J)$. We apply induction on the length $l(w)$.

If $l(w) = 0$ then result is trivial. If $l(w) = 1$ then $w = s_{\alpha_i}$ for some $\alpha_i \in \Pi$. If $i \in J$, then we are done. If $i \notin J$, then α_i is linearly independent of $\{\alpha_j \mid j \in J\}$. But

$$w\lambda = s_{\alpha_i} \lambda = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i = \lambda - \sum_{j \in J} a_j \alpha_j.$$

So, J must be empty, and then $W_J = 1$. Therefore, $w\lambda = \lambda$.

Assume that $l(w) > 1$. Let $w^J = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced expression. If $w^J = 1$ then we are done. If $w^J \neq 1$ then $r \geq 1$. Then $w^J \alpha_r < 0$ by [7, Section 4.3, Theorem B] and $r \notin J$. Write $w_J \lambda = \lambda - \sum_{i \in J} b_i \alpha_i$ where $b_i \geq 0$ for all $i \in J$. We have

$$\begin{aligned} (w\lambda, w^J \alpha_r) &= (w^J w_J \lambda, w^J \alpha_r) = (w_J \lambda, \alpha_r) \\ &= (\lambda - \sum_{i \in J} b_i \alpha_i, \alpha_r) \geq 0, \end{aligned}$$

since $r \notin J$ and $(\alpha_i, \alpha_r) \leq 0$ for $i \in J$. Divide the discussion into two cases.

Case 1: $(w\lambda, w^J \alpha_r) = 0$. We have

$$(\lambda, w_J^{-1} \alpha_r) = (w_J \lambda, \alpha_r) = 0.$$

It follows that

$$s_{w_J^{-1} \alpha_r} \lambda = w_J^{-1} s_{\alpha_r} w_J \lambda = \lambda,$$

and so $s_{\alpha_r} w_J \lambda = w_J \lambda$. Hence,

$$w\lambda = w^J w_J \lambda = s_{\alpha_1} \dots s_{\alpha_{r-1}} w_J \lambda.$$

But $l(s_{\alpha_1} \dots s_{\alpha_{r-1}} w_J) < l(w)$. From the induction hypothesis, there exists an element $u \in W_J$ such that $s_{\alpha_1} \dots s_{\alpha_{r-1}} w_J \lambda = u\lambda$. Thus, $w\lambda = u\lambda$.

Case 2: $(w\lambda, w^J \alpha_r) > 0$. Now, $(\lambda - \sum_{j \in J} a_j \alpha_j, w^J \alpha_r) > 0$. Then,

$$\mu = \lambda - \sum_{j \in J} a_j \alpha_j - w^J \alpha_r$$

is a weight by Proposition 2.1. Since $\mu \leq \lambda$ and $w^J \alpha_r$ is a negative root, $w^J \alpha_r$ has to be linear combination of α_j 's for $j \in J$. Write $\gamma = w^J \alpha_r$. Then $w^J s_{\alpha_r} = s_\gamma w^J$ and

$$s_\gamma w\lambda = w^J s_{\alpha_r} w_J \lambda = \lambda - \sum_{j \in J} c_j \alpha_j$$

where $c_j \geq 0$. Since $l(w^J s_{\alpha_r} w_J) < l(w)$, by the induction hypothesis, there exists $v \in W_J$ such that $w^J s_{\alpha_r} w_J \lambda = s_\gamma w \lambda = v \lambda$. Let $u = s_\gamma v \in W_J$. Therefore, $w \lambda = s_\gamma v \lambda = u \lambda$. This completes the proof. \square

For a subset Σ of $\{-\lambda\} \cup \Pi$, denote by $\Gamma(\Sigma)$ the subdiagram of the extended Dynkin diagram having Σ as vertices, and $\Gamma^0(\Sigma)$ the connected component of $\Gamma(\Sigma)$ containing $\{-\lambda\}$. Denote by V_I the set of vertices of $\Gamma^0(\Pi_{\bar{I}} \cup \{-\lambda\})$. Let $\Pi_{\bar{I}^0} = \Pi \cap V_I$ and let \bar{I}^0 be the set of indices of the vertices in $\Pi_{\bar{I}^0}$. Let E_{F_I} be the space spanned by $\{\mu - \nu \mid \mu, \nu \in F_I\}$.

Theorem 4.3 *Let $I \subseteq \mathbf{n}$ and u_0 be the longest element in $W_{\bar{I}^0}$. If F_I is a standard parabolic face of \mathbf{P} , then*

- (1) F_I is the convex hull of $W_{\bar{I}^0} \lambda$,
- (2) $u_0 \lambda$ is the least element in P_I , and
- (3) E_{F_I} is spanned by the simple roots in $\Pi_{\bar{I}^0}$ and $\dim E_{F_I} = |\bar{I}^0|$.

Proof. For (1), let $w \in W_{\bar{I}^0}$. Then $w \lambda = \lambda - \sum_{j \in \bar{I}^0} a_j \alpha_j \in P_I$, since $\bar{I}^0 \cap I = \emptyset$. Notice that P_I is the set of weights in F_I . We obtain that $w \lambda \in F_I$, and the convex hull of $W_{\bar{I}^0} \lambda$ is contained in F_I .

Conversely, for any $w \in W$ assume that $w \lambda = \lambda - \sum_{j \in J} a_j \alpha_j$ with $a_j > 0$. Then $J \subseteq \bar{I}$, since $c_i(w \lambda) = c_i(\lambda) = m_i$ for all $i \in I$. By Proposition 4.2, there exists an element $u \in W_J$ such that $u \lambda = w \lambda$. If $\Pi_J \cup \{-\lambda\}$ is connected in the extended Dynkin diagram, then we are done. Otherwise, we can assume that $J = J' \cup \bar{I}^0$. Notice that $\Pi_{\bar{I}^0} \cup \{-\lambda\}$ is a connected component in the Dynkin diagram. Then $W_J \cong W_{\bar{I}^0} \times W_{J'}$. Write $u = xy$, where $x \in W_{\bar{I}^0}$ and $y \in W_{J'}$. Thus, $w \lambda = u \lambda = xy \lambda = x \lambda$ since $(\lambda, \alpha_j) = 0$ for all $j \in J'$. In other words, $w \lambda \in W_{\bar{I}^0} \lambda$. It follows that F_I is included in the convex hull of $W_{\bar{I}^0} \lambda$. This completes the proof of (1).

To prove (2), it suffices to show that $u_0 \lambda < w \lambda$ for all $w \in W_{\bar{I}^0}$, since other weights in P_I are linear combinations of $W_{\bar{I}^0} \lambda$ with non-negative coefficients whose sum is equal to 1. Thanks to $\lambda \geq u_0^{-1} w \lambda$, we have $\lambda - u_0^{-1} w \lambda = \sum_{j \in \bar{I}^0} a_j \alpha_j$ with $a_j \geq 0$. It follows that

$$\begin{aligned} u_0 \lambda - w \lambda &= u_0(\lambda - u_0^{-1} w \lambda) \\ &= - \sum_{j \in \bar{I}^0} a_j (-u_0(\alpha_j)) \end{aligned}$$

where $-u_0(\alpha_j) \in \Pi_{\bar{I}^0}$ since $u_0 \Pi_{\bar{I}^0} = -\Pi_{\bar{I}^0}$. So $u_0 \lambda \leq w \lambda$. This proves (2).

We now prove (3). If $u_0\lambda = \lambda$, then λ is the least element of $W_{\bar{T}^0}\lambda$ by (2) and $W_{\bar{T}^0}\lambda = \{\lambda\}$. It follows that $\Pi_{\bar{T}^0} = \emptyset$ and $F_I = \{\lambda\}$. If $u_0\lambda < \lambda$, let

$$\lambda - u_0\lambda = \sum_{j \in \bar{T}^0} a_j \alpha_j. \quad (1)$$

By Lemma 4.1, there are simple roots $\alpha_{i_1}, \dots, \alpha_{i_k}$ such that $u_0\lambda + \alpha_{i_1} + \dots + \alpha_{i_s}$ are weights in F_I for $s = 1, \dots, k$. Any element $\lambda - w\lambda$ is a linear combination of $\alpha_{i_1}, \dots, \alpha_{i_k}$, for $w \in W_{\bar{T}^0}\lambda$. Hence $\alpha_{i_1}, \dots, \alpha_{i_k}$ span E_{F_I} . But $\alpha_{i_s} \in \Pi_{\bar{T}^0}$ by (1). The dimension of F_I is the number of nonzero terms on the right-hand side of (1). We claim that $a_j > 0$ for all $j \in \bar{T}^0$. To this end, for any $\alpha_j \in \Pi_{\bar{T}^0}$, we can find a shortest connected path: $\alpha_j, \alpha_{j_1}, \dots, \alpha_{j_m}, \{-\lambda\}$, from α_j to $\{-\lambda\}$ in the extended Dynkin diagram. Then the coefficient of α_j in $\lambda - s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_m}} \lambda$ is positive. Hence $a_j > 0$ in (1) for $j \in \bar{T}^0$, since $s_{\alpha_j} s_{\alpha_{j_1}} \dots s_{\alpha_{j_m}} \lambda \geq u_0\lambda$. This completes the proof. \square

Note that for different subsets $I, J \subseteq \mathbf{n}$, it is possible that $F_I = F_J$.

Corollary 4.4 *For any two subsets $I, J \subseteq \mathbf{n}$, $F_I = F_J$ if and only if $\Pi_{\bar{T}^0} = \Pi_{\bar{J}^0}$.*

Proof. (\Rightarrow) Use η_I to denote the least element of F_I . If $F_I = F_J$, then they have the same least element $\eta_I (= \eta_J)$. By the proof of (3) in Theorem 4.3,

$$\lambda - \eta_I = \sum_{i \in \bar{T}^0} a_i \alpha_i \quad \text{and} \quad \lambda - \eta_J = \sum_{i \in \bar{J}^0} b_i \alpha_i,$$

where $a_i > 0$ for all $i \in \bar{T}^0$, and $b_i > 0$ for all $i \in \bar{J}^0$. It follows that $\Pi_{\bar{T}^0} = \Pi_{\bar{J}^0}$.

(\Leftarrow) This is trivial by (1) of Theorem 4.3. \square

The theorem below follows from Theorem 4.3 and Corollary 4.4.

Theorem 4.5 *The standard parabolic faces of \mathbf{P} are in bijection with the connected subdiagrams of the extended Dynkin diagram that contain $\{-\lambda\}$. This bijection is an isomorphism of posets with respect to inclusion.*

This theorem is the Lie algebra version of Theorem C, which is a consequence of [15, Theorem 4.16] and [8, Theorem 2] and [13, Theorem 8.4] and [16, Section 5]. In fact, the bijection can be explicitly described as: $F_I \rightarrow \bar{T}^0$.

Corollary 4.6 *All coordinate faces are distinct.*

Proof. Let η_i be the least element of F_i . We have $(\eta_i, \alpha_j) \leq 0$ for all $j \neq i$. Otherwise $\eta_i - \alpha_j$ is a weight in $P_i \subseteq F_i$ by Proposition 2.1 and η_i would not be minimal. This yields $(\eta_i, \alpha_i) > 0$ since $(\eta_i, \eta_i) > 0$. Thus $\eta_i \neq \eta_j$, and hence $P_i \neq P_j$ if $i \neq j$. The coordinate faces F_i and F_j are then distinct since P_i is exactly the set of weights in F_i for all $i \in \mathbf{n}$. \square

The following is a criterion that tests whether a coordinate face is a facet.

Corollary 4.7 *Let F_i be a coordinate face and η_i be the least element in F_i . Then F_i is a facet if and only if $(\eta_i, \tilde{\omega}_j) \neq m_j$ for all $j \neq i$.*

Proof. (\Rightarrow) By Theorem 4.3, if F_i is a facet then $\dim F_i = |\Pi_{T^0}| = n - 1$ where $I = \{i\}$. This implies that $\Pi_{T^0} = \Pi \setminus \{i\}$ and $\eta_i = \lambda - \sum_{\alpha_j \in \Pi_{T^0}} a_j \alpha_j$ with $a_j > 0$ for all $j \neq i$. Therefore, $(\eta_i, \tilde{\omega}_j) \neq m_j$ for all $j \neq i$.

(\Leftarrow) If $(\eta_i, \tilde{\omega}_j) = m_j - a_j \neq m_j$ for all $j \neq i$, then $a_j > 0$ for all $j \neq i$. By Lemma 4.1 or the proof of Theorem 4.3, E_{F_i} is spanned by $\{\alpha_j \mid j \in \mathbf{n} \setminus \{i\}\}$. Hence $\dim F_i = n - 1$. \square

Lemma 4.8 *Let α be a root in Φ and μ be a weight in $P(\lambda)$ and denote the α -string through μ by $\alpha - (\mu)$. Then*

$$\sum_{\gamma \in \alpha - (\mu)} (\gamma, \alpha) = 0.$$

Proof. Without loss of generality, we assume that μ is the start of its α -string. Then the weight string is

$$\mu, \mu + \alpha, \dots, \mu + q\alpha, \quad \text{where } q = -\frac{2(\mu, \alpha)}{(\alpha, \alpha)}.$$

If q is even, the middle weight $\mu - \frac{(\mu, \alpha)}{(\alpha, \alpha)}\alpha$ is orthogonal to α , and the sum of the two symmetric weights about the middle weight is orthogonal to α . If q is odd, the sum of the two symmetric weights is orthogonal to α . \square

Proposition 4.9 *The barycenter of the weights in the i th coordinate face F_i is parallel to the i th fundamental weight, and*

$$\sum_{\mu \in P_i} \mu = \frac{m_i |P_i|}{(\tilde{\omega}_i, \tilde{\omega}_i)} \tilde{\omega}_i.$$

Proof. For each $j \in \mathbf{n} \setminus \{i\}$, P_i is a disjoint union of α_j -strings. It follows from Lemma 4.8 that

$$\sum_{\mu \in P_i} (\mu, \alpha_j) = 0.$$

Hence, $\sum_{\mu \in P_i} \mu = a \tilde{\omega}_i$. In view of $(\mu, \tilde{\omega}_i) = m_i$ for all $\mu \in P_i$, we have

$$(a \tilde{\omega}_i, \tilde{\omega}_i) = \left(\sum_{\mu \in P_i} \mu, \tilde{\omega}_i \right) = |P_i|(\mu, \tilde{\omega}_i) = m_i |P_i|.$$

The desired result follows. \square

Lemma 4.10 *Let $I \subseteq \mathbf{n}$. Then the barycenter of the standard parabolic face F_I is $\sum_{i \in I} a_i \tilde{\omega}_i$ where a_i are non-negative rational numbers for all $i \in I$.*

Proof. As P_I is a disjoint union of α_j -strings for any fixed $j \in \mathbf{n} \setminus I$, we know that $(\sum_{\mu \in P_I} \mu, \alpha_j) = 0$ for all $j \notin I$. Thus, $\sum_{\mu \in P_I} \mu = \sum_{i \in I} a_i \tilde{\omega}_i$. For any $j \in I$, we have

$$\sum_{i \in I} a_i (\tilde{\omega}_i, \tilde{\omega}_j) = \left(\sum_{\mu \in P_I} \mu, \tilde{\omega}_j \right) = |P_I|(\mu, \tilde{\omega}_j).$$

For $i, j \in I$ and $\mu \in P_I$, the numbers $(\tilde{\omega}_i, \tilde{\omega}_j)$ and $(\mu, \tilde{\omega}_j)$ are rational numbers, so are all a_i 's. We claim that $(\mu, \alpha_i) \geq 0$ for $\mu \in P_I$. If not, it follows from $(\mu, \alpha_i) < 0$ that $\mu + \alpha_i \in P(\lambda)$, which shows that $c_i(\mu + \alpha_i) = m_i + 1$. This contradicts that $\mu \leq \lambda$. Thus $a_i \geq 0$, as desired. \square

By the lemma above, the barycenter of a standard parabolic face is in the closure of the fundamental Weyl chamber [5]. Since the Weyl group acts on the fundamental chamber transitively, we have

Corollary 4.11 *Two distinct standard parabolic faces of the weight polytope \mathbf{P} cannot be transformed into one another by elements of the Weyl group.*

Lemma 4.12 *Let \mathbf{P} be a polytope (or cone). Let ν be a fixed vertex. Suppose $[\mu, \nu]$ is an edge (or spans an edge in the cone) with*

$$\mu - \nu = a_1(x_1 - \nu) + \cdots + a_n(x_n - \nu)$$

where $a_i > 0$ and $x_i \in \mathbf{P}$. Then x_i are in the same edge that contains $[\mu, \nu]$.

Proof. Assume that $[\mu, \nu]$ (or its span for the cone) is an intersection of facet F_k ($k = 1, \dots, r$). Assume that the linear equation for the hyperplane containing F_k is $L_k(x) = 0$ for $k \in [r]$. We want to prove that $L_k(x_i) = L_k(\nu)$ for all $k \in [r]$ and $i \in \mathbf{n}$. We have

$$L_k(\mu - \nu) = a_1(L_k(x_1) - L_k(\nu)) + \cdots + a_n(L_k(x_n) - L_k(\nu)) = 0.$$

Since $x_i \in \mathbf{P}$, $L_k(x_i) - L_k(\nu)$ have the same sign for all x_i ($i = 1, \dots, n$). Therefore, $L_k(x_i) = L_k(\nu)$ for all $k \in [r]$. This means that x_i is in the same edge that contains $[\mu, \nu]$. \square

Lemma 4.13 *Every edge of \mathbf{P} is parallel to a root.*

Proof. It suffices to prove this for edges attached to the highest dominant weight λ . Assume that $\mu = w\lambda \neq \lambda$ and the segment $[\lambda, \mu]$ is an edge. We assert that μ and λ are separated by one and only one reflection hyperplane H_α where α is a positive root and $H_\alpha = \{x \in E \mid (x, \alpha) = 0\}$.

If not, there are two positive distinct roots α, β such that μ and λ are separated by two different reflection hyperplanes H_α and H_β . Then $(\lambda, \alpha) > 0$, $(\mu, \alpha) < 0$ and $(\lambda, \beta) > 0$, $(\mu, \beta) < 0$. Therefore, $\lambda - \alpha, \lambda - \beta, \mu + \alpha$, and $\mu + \beta$ are weights in $P(\lambda)$. We have

$$\mu - \lambda = \frac{(\mu + \alpha - \lambda) + (\mu + \beta - \lambda) + (\lambda - \alpha - \lambda) + (\lambda - \beta - \lambda)}{2},$$

which is a contradiction by Lemma 4.12. Thus μ and λ are separated by one hyperplane H_α . But μ and λ are in the closures of two different Weyl chambers, one of which is transformed to the other by the reflection s_α . It follows that $\mu = s_\alpha \lambda$ from the fact that the Weyl group acts on all the chambers simply transitively. Here α is a positive root but not necessary a simple root. \square

The following theorem follows directly from the above lemma.

Theorem 4.14 *Given a face F of \mathbf{P} , let $E_F = \text{Span}\{\mu - \nu \mid \mu, \nu \in F\}$. Then the space E_F is spanned by roots for any face F of \mathbf{P} .*

Now we are ready to show that all nonempty faces of a weight polytope are parabolic. The proof of the following theorem is patented after that of [3, Theorem 5.6]. For completeness, we give the proof here.

Theorem 4.15 *Every nonempty face of a weight polytope is parabolic.*

Proof. Let F be a face with $\dim F = n - p$ where $1 \leq p \leq n$. We use induction on p to prove the desired result. If $p = 1$ then F is a facet. From Theorem 4.14 and Proposition 2.2 it follows that $\Phi \cap E_F$ is a root subsystem of Φ with rank $n - 1$. Let Π' be a root basis of $\Phi \cap E_F$. Then there exists $w \in W$ such that $w\Pi' \subseteq \Pi$. Let α_i be the only fundamental root in Π which does not belong to $w\Pi'$. So wE_F is perpendicular to $\tilde{\omega}_i$, denoted by $wE_F \perp \tilde{\omega}_i$. Therefore, there exists a real number a such that $(x, \tilde{\omega}_i) = a$ for all $x \in wF$. It follows that $\{x \in E \mid (x, \tilde{\omega}_i) = a\}$ is a supporting hyperplane of the weight polytope. Let $l_i = (w_0\lambda, \tilde{\omega}_i)$ where w_0 is the longest element in the Weyl group W . Then $l_i \leq a \leq m_i$. This forces $a = m_i$ or $a = l_i$. If not, vertices λ and $-w_0\lambda$ will be at different sides of the hyperplane $\{x \in E \mid (x, \tilde{\omega}_i) = a\}$, a contradiction. If $a = m_i$ then $wF = F_i$. If $a = l_i$, then $-l_i = (\lambda, \tilde{\omega}_{i'}) = m_{i'}$ where $\tilde{\omega}_{i'} = -w_0(\tilde{\omega}_i)$. Hence $w_0wF = F_{i'}$.

Now we assume that $n \geq p > 1$ and all faces of the weight polytope \mathbf{P} of dimension greater than $n - p$ are parabolic. By the induction hypothesis, without loss of generality, we can assume that F is a facet of a standard parabolic face F_I where $\dim F = n - p$ and $\dim F_I = n - p + 1$. By Theorem 4.3, F_I is the convex hull of $W_{\bar{T}^0}\lambda$ and F_I is spanned by $\{\alpha_i \mid i \in \bar{T}^0\}$. Note that there exists $w \in W_{\bar{T}^0}$ such that $wE_F \perp \tilde{\omega}_j$ where $j \in \bar{T}^0$. Meanwhile, there exists a real number a such that $(x, \tilde{\omega}_j) = a$ for all $x \in wF$. Let $l_j = (\eta_I, \tilde{\omega}_j)$. Recall $\eta_I = u_0\lambda$ is the least element in F_I where u_0 is the longest element in $W_{\bar{T}^0}$. Then $a = l_j$ or $a = m_j$.

If $a = m_j$ then $\lambda \in F$. Hence $wF = \{\lambda + \mu \mid \mu \in E_{F_I} \cap \tilde{\omega}_j^\perp\} \cap F_I = F_I \cap F_j$. We are done. If $a = l_j$, then $\eta_I \in wF$. Hence $wF = \{\eta_I + \mu \mid \mu \in E_{F_I} \cap \tilde{\omega}_j^\perp\} \cap F_I$. Let π be the orthogonal projection of E onto E_{F_I} . Then $E_{F_I} \cap \tilde{\omega}_k^\perp = E_{F_I} \cap \pi(\tilde{\omega}_k)^\perp$ for all $k \in \bar{T}^0$. Hence $\{\pi(\tilde{\omega}_k) \mid k \in \bar{T}^0\}$ is the coweights with respect to $\Pi_{\bar{T}^0}$ in root system $\Phi(\Pi_{\bar{T}^0})$. Thus, there exists j' such that $u_0(\pi(\tilde{\omega}_j)) = -\pi(\tilde{\omega}_{j'})$. Since $u_0(F_I) = F_I$, we have $u_0(E_{F_I} \cap \tilde{\omega}_j^\perp) = E_{F_I} \cap \pi(\tilde{\omega}_{j'})^\perp = E_{F_I} \cap \tilde{\omega}_{j'}^\perp$. Thus, F is parabolic, since $u_0wF = \{u_0(\eta_I) + \mu \mid \mu \in E_{F_I} \cap \tilde{\omega}_{j'}^\perp\} \cap F_I = \{\lambda + \mu \mid \mu \in E_{F_I} \cap \tilde{\omega}_{j'}^\perp\} \cap F_I = F_I \cap F_{j'}$. \square

We conclude this section by giving the f -polynomial of weight polytopes.

Theorem 4.16 *Let $\mathcal{I} = \{I \subseteq \mathbf{n} \mid \Gamma(\Pi_I \cup \{-\lambda\}) \text{ is connected}\}$ and let J be the type of λ . For each $I \in \mathcal{I}$, let $I^* = I \cup \{j \in J \mid (\alpha_j, \alpha_i) = 0 \text{ for all } i \in I\}$. Then the f -polynomial of \mathbf{P} is*

$$\sum_{I \in \mathcal{I}} \frac{|W|}{|W_{I^*}|} t^{|I|}.$$

Proof. Consider the action of the Weyl group W on the faces of \mathbf{P} . A simple calculation yields that W_{I^*} is the stabilizer of the standard parabolic face F , where F is the convex hull of $W_I\lambda$. \square

Acknowledgements The authors thank Dr. Reginald Koo for his careful reading of and comments on the paper.

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